Exercise 35

If the minute hand of a clock has length r (in centimeters), find the rate at which it sweeps out area as a function of r.

Solution

Start with the formula for the area of a circular sector.

$$A = \frac{1}{2}r^2\theta$$

Take the derivative of both sides with respect to t by using the chain and product rules.

$$\begin{aligned}
\frac{d}{dt}(A) &= \frac{d}{dt}\left(\frac{1}{2}r^2\theta\right) \\
\frac{dA}{dt} &= \frac{1}{2}\left\{\left[\frac{d}{dt}(r^2)\right]\theta + r^2\left[\frac{d}{dt}(\theta)\right]\right\} \\
&= \frac{1}{2}\left[\left(2r\cdot\frac{dr}{dt}\right)\theta + r^2\left(\frac{d\theta}{dt}\right)\right] \\
&= r\frac{dr}{dt}\theta + \frac{1}{2}r^2\frac{d\theta}{dt}
\end{aligned}$$

The minute hand doesn't change in length, so $dr/dt = 0$. Also, the minute hand moves through 2π radians every hour, so $d\theta/dt = 2\pi/60$ rad/min. Therefore,

$$\frac{dA}{dt} = r(0)\theta + \frac{1}{2}r^2\left(\frac{2\pi}{60}\right) = \frac{\pi r^2}{60}\ \frac{\text{cm}^2}{\text{min}}.$$